Consider a classical control system as it was defined by Pontryagin:

\[ \dot{x} = f(x, u), \quad x \in M, \; u \in U. \]  

(1)

Assume that the state space \( M \) is a smooth manifold, the set of control parameters \( U \) is a closed subset of another smooth manifold, the right-hand side \( f \) is smooth, and a reasonable completeness assumption allows to extend solutions of ordinary differential equations to the whole time axis.

We call controls the mappings \( u: (t, x) \mapsto u(t, x) \) with values in \( U \) that are smooth with respect to \( x \) and measurable bounded with respect to \( t \): a mixture of the program and feedback controls. Now plug-in a control in system (1) and obtain a time-varying ordinary differential equation

\[ \dot{x} = f(x, u(t, x)), \]  

(2)

which generates a family of diffeomorphisms \( P_t: M \to M \), where \( P_0(x) = x \) ands the curves \( t \mapsto P_t(x) \) satisfy (2) for any \( x \in M \). We say that \( t \mapsto P_t \) is an admissible “trajectory” in the group of diffeomorphisms associated to the control \( u \).

Given an integral cost functional

\[ J(u(\cdot)) = \int_0^T \varphi(x(t), u(t)) \, dt \]

and a probability measure \( \mu \) on \( M \), we set

\[ J_\mu(u) = \int_0^T \int_M \varphi(P_t(x), u(t, x)) \, d\mu dt \]

a functional on the space of controls \( u \).

In my talk, I am going to discuss the controllability and optimal control issues for the defined in this way systems on the group of diffeomorphisms.