Control of Diffeomorphisms and Densities

Andrei A. Agrachev

SISSA, Trieste, Italy and Steklov Mathematical Institute, Moscow, Russia
e-mail: agrachev@sissa.it

Consider a classical control system as it was defined by Pontryagin:

$$\dot{x} = f(x, u), \quad x \in M, \ u \in U. \tag{1}$$

Assume that the state space M is a smooth manifold, the set of control parameters U is a closed subset of another smooth manifold, the right-hand side f is smooth, and a reasonable completeness assumption allows to extend solutions of ordinary differential equations to the whole time axis.

We call controls the mappings $\mathbf{u}\colon (t, x) \mapsto \mathbf{u}(t, x)$ with values in U that are smooth with respect to x and measurable bounded with respect to t: a mixture of the program and feedback controls. Now plug-in a control in system (1) and obtain a time-varying ordinary differential equation

$$\dot{x} = f(x, \mathbf{u}(t, x)), \tag{2}$$

which generates a family of diffeomorphisms $P_t\colon M \to M$, where $P_0(x) = x$ ands the curves $t \mapsto P_t(x)$ satisfy (2) for any $x \in M$. We say that $t \mapsto P_t$ is an admissible "trajectory" in the group of diffeomorphisms associated to the control $\mathbf{u}$.

Given an integral cost functional

$$J(u(\cdot)) = \int_0^T \varphi(x(t), u(t))\, dt$$

and a probability measure μ on M, we set

$$\mathbf{J}_\mu(\mathbf{u}) = \int_0^T \int_M \varphi(P_t(x), \mathbf{u}(t, x))\, d\mu dt$$

a functional on the space of controls $\mathbf{u}$.

In my talk, I am going to discuss the controllabilty and optimal control issues for the defined in this way systems on the group of diffeomorphisms.