Hamiltonian format of Pontryagin's maximum principle

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1. Hamiltonian format is native for the maximum principle regardless of any special restrictions imposed on the control equations of the optimal problem under consideration. It assigns canonically to any optimal problem a Hamiltonian system with parameters, complemented with the maximum condition, which "dynamically" eliminates the parameters in the process of solving the initial value problem for the Hamiltonian system as we proceed along the trajectory. Thus the extremals of the problem are generated as simultaneous solutions of a regular Hamiltonian system with parameters and the maximum condition (to which all singularities of the problem are "relegated"), and not as a Hamiltonian flow, i.e. as a family of solutions of an initial value problem of a Hamiltonian system (without parameters), to which the Euler-Lagrange equation could be reduced in the *regular* case of the classical calculus of variations.

2. I shall give here an invariant formulation of the Hamiltonian format of the maximum principle for the time-optimal problem.

   Let the control system be given by the equation

$$\frac{dx}{dt} = X(x,u) = X,\ u \in U,$$

   where X is a vector field on the configuration space M, $x \in M$, u is the control parameter, U is the set of admissible values of u.

   With the vector field X we canonically associate a scalar-valued fiber-wise linear function H_X on the cotangent bundle T^*M,

$$H_X(\xi,u) \stackrel{def}{=} \langle \xi, X(\pi\xi,u)\rangle,\ \xi \in T^*M,\ u \in U,$$

   where $\pi\colon T^*M \longrightarrow M$ is the canonical projection. Hence a family of Hamiltonian vector fields $\overrightarrow{H}_X$ defined by the family of Hamiltonians H_X canonically corresponds to the time-optimal problem. Every initial value $\xi \in T^*M$, $\xi \notin M$, defines an extremal of the time optimal problem as a trajectory $\xi(t)$, $\xi(0) = \xi$, of the Hamiltonian vector field $\overrightarrow{H}_X$, from which the parameter $u \in U$ is "dynamically" eliminated by the maximum condition

$$H_X(\xi(t), u(t)) = \max_{v \in U} H_X(\xi(t), v).$$

Every solution of the optimal problem could be obtained in the described way.

3. Since the time-optimal problem is completely defined by the vector field $X(x, u)$, it is natural to expect that every first order "infinitesimal object" invariantly connected with our problem should be canonically (tensorially) expressed through the differential of the flow e^{tX}. To give this expression in our case, we first remark that the Hamiltonian vector field $\overrightarrow{H}_X$ coincides with the vector field ad_X on the cotangent bundle T^*M, which is uniquely defined by the equations

$$ad_X a = Xa \quad \forall a \in C^\infty(M), \;\; ad_X Y = [X, Y] \quad \forall Y \in Vect\, M.$$

Let $\mathcal{L}_X$ be the Lie derivative over the field X, $e^{t\mathcal{L}_X} = e_*^{tX}$, where e_*^{tX} is the differential of the flow e^{tX} on M.

According to the existing duality between the flows $e^{t\mathcal{L}_X}$ and e^{tad_X} expressed by the identity

$$e^{tX}\langle \omega, X\rangle = \langle e^{t\mathcal{L}_X}\omega, e^{tad_X}Y\rangle \quad \forall Y \in Vect\, M, \;\; \omega \in \Lambda^{(1)}(M),$$

we have

$$e^{tad_X} = \left(e^{-t\mathcal{L}_X}\right)^* = \left(e^{t\mathcal{L}_X}\right)^{*-1}.$$

Hence the flow generated by the Hamiltonian vector field $\overrightarrow{H}_X = ad_X$ is inverse to the conjugate of the differential $e_*^{tX} = e^{t\mathcal{L}_X}$, in particular, it is a bundle isomorphism of the cotangent bundle T^*M over the flow e^{tX} for $\forall t$, and the vector field $\overrightarrow{H}_X$ is a "Hamiltonian lift" of the vector field X.

Differentiating the above identity with respect to t we establish the "infinitesimal" duality between $\mathcal{L}_X$ and ad_X (the generalized Leibnitz rule),

$$X\langle \omega, Y\rangle = \langle \mathcal{L}_X\omega, Y\rangle + \langle \omega, ad_X Y\rangle \;\forall Y \in Vect\, M, \;\; \omega \in \Lambda^{(1)}(M).$$

The indicated relations completely identify the Hamiltonian vector field $\overrightarrow{H}_X$, hence the Hamiltonian format of the maximum principle.

4. Whereas the Lie derivative $\mathcal{L}_X$ and the flow it generates on the tangent bundle TM, the differential $e_*^{tX} = e^{t\mathcal{L}_X}$, are, in one or another

way, the objects of everyday mathematical practice, the dual vector field to $\mathcal{L}_X$, the Hamiltonian lift $\overrightarrow{H_X} = ad_X$ on the cotangent bundle T^*M, and the corresponding flow e^{tad_X} were first introduced for computational purposes only in 1956 by L. S. Pontryagin under the name of "conjugate system" , and through the discovery of the maximum principle became since then a standard computational tool in engineering practice. Today, they are absolutely inevitable in optimization problems related to trajectory variations.

I think, it would be historically justified to baptize the vector field ad_X, (considered precisely as a vector field on T^*M, and not as a derivation on the $C^\infty(M)$-module of vector fields on M), as the *Pontryagin derivative.*