The Thomson theorem on instability: its topological meaning and generalizations

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According to the classical Thomson theorem if the inertia index of the total energy in the state of an isolated equilibrium is odd then this equilibrium is unstable. This theorem establishes obstacles for gyroscopic stabilization of unstable equilibriums because gyroscopic forces do not change the total energy. By the way, in the theory of gyroscopic stabilization an essential role is played by the Pontryagin space (a Hilbert space with an indefinite metric). In the talk I will discuss new results in the problem of gyroscopic stabilization. These results are based in particular, on the search for vortex (non-Lagrangian) invariant manifolds of Hamiltonian equations.

The Thomson theorem can be formulated for generic systems of differential equations which admit non-increasing functions with Morse critical points. Moreover, in many cases it is possible to deal with non-isolated equilibriums. A topological proof of this statement is based on the Poincare theorem on the sum of indices of singular points of a vector field.

I will also discuss a more complicated problem on obstacles to gyroscopic stabilization when equilibrium positions are degenerate points of the potential energy. General theorems on instability are based on the search for solutions asymptotic to the equilibrium in the form of series in inverse degrees of the time variable with coefficients, polynomially depending on logarithm of the time. On the other hand, possibility of constructing of such series is closely connected with the topology of some real algebraic varieties (more precisely, with an estimate of their Euler characteristic). Some general results imply an extension of the Thomson theorem to degenerate equilibriums, which appear unavoidably in typical families of mechanical systems, depending on 5 or less parameters.